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An Alternative Approach to Obtain the Exact Wave Functions of Time-Dependent Hamiltonian Systems Involving Quadratic, Inverse Quadratic, and $(1/x)p + p(1/x)$ Terms

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By using the Lewis-Riesenfeld theory and algebraic method, we present an alternative approach to obtain the exact solution of time-dependent Hamiltonian systems involving quadratic, inverse quadratic and $(1/x)p + p(1/x)$ terms. This solution is discussed and compared with that obtained by Choi, J. R. (2003). [*International Journal of Theoretical Physics* **42**, 853].

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In the last few decades the problem of time-dependent systems have played a major role in the study of several physics phenomena. A great deal of attention has been paid to some specific problems of time-dependent oscillators among them the time-dependent singular oscillator (Calogero, 1969; Calogero, 1971; Malkin and Man'ko, 1972; Dodonov *et al.*, 1974; Markov, 1989; Maamache, 1995, 1996; Pedrosa *et al.*, 1997; Dodonov *et al.*, 1998; Trifonov, 1999; Maamache *et al.*, 1999a,b; Um *et al.*, 2002; Maamache and Chourti, 2000; Maamache and Bekkar, 2003; Pedrosa and Guedes, 2003; Yüce, 2003). In fact this specific problem has been studied extensively in different directions by many authors who obtained closed-form solutions in explicit form . The construction of the invariant (constants of the motion), which has attracted much attention describes a quantum system governed by a time-dependent Hamiltonian. Lewis and Riensenfeld (Lewis and Riesenfeld, 1969) have shown that, if the system admits an invariant $I(t)$, it is

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possible to find a privileged basis of eigenstates of this operator, which when multiplied by suitable time-dependent phase factor, evolve according to the time-dependent Schrödinger equation.

In a recent paper (Choi, 2003), the quantum time-dependent Hamiltonian systems involving quadratic, inverse quadratic and $(1/x)p + p(1/x)$ terms has been considered. The term containing $(1/x)p + p(1/x)$ gives the expression containing $\frac{1}{x}\frac{\partial}{\partial x}$ in coordinate space that appears in radial equation of quantum many body problems (Calogero, 1969; Um *et al.*, 2002).

In this paper, we shall derive the correct wave function of a time-dependent Hamiltonian system involving quadratic, inverse quadratic and $(1/x)p + p(1/x)$ terms using the Lie algebraic approach previously developed (Maamache, 1995).

Let us consider the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = H(t)\psi(x,t) \tag{1}$$

associated to the Hamiltonian defined in (Choi, 2003)

$$H(x,p,t) = A(t)p^2 + B(t)(xp+px) + D(t)x^2 + \frac{E(t)}{x^2} + C(t)\left(\frac{1}{x}p + p\frac{1}{x}\right) \tag{2}$$

where A(t) B(t),C(t), D(t), E(t) are time-dependent coefficients, and x and p are the canonical coordinates.

The ratio $\frac{E(t)}{A(t)}$ and $\frac{C(t)}{A(t)}$ are constant, this fact can be easily inferred from Equations (7)–(9), (11) and (13) of Ref. Choi (2003). Also, this can be explained by the fact that that its Hamiltonian is a linear combination of generators of the SU(1,1) algebra.

Note that the fact that the ratio $\frac{E(t)}{A(t)}$ is constant, does not have to be imposed as stated in Ref. Choi (2003).

For the construction of an exact invariant for the quantum system described by the time-dependent Hamiltonian (2), we perform the time-dependent unitary transformation

$$\psi(x,t) = U(t)\varphi(x,t) \tag{3}$$

where

$$U(t) = \exp\left[\frac{i}{\hbar}\int\frac{2C(t)B(t)}{A(t)}dt\right] \tag{4}$$

Under this unitary transformation the Schrödinger Equation (1) is mapped into

$$i\hbar\frac{\partial}{\partial t}\varphi(x,t) = \widetilde{H}(x,p,t)\varphi(x,t) \tag{5}$$

where the new Hamiltonian $\widetilde{H}(x, p, t)$ as a linear combination of the Hermitian basis, i.e.

$$\widetilde{H}(x, p, t) = A(t)T_1 + B(t)T_2 + D(t)T_3 \tag{6}$$

where

$$\begin{aligned} T_1 &= p^2 + \frac{E}{A}\frac{1}{x^2} + \frac{C}{A}\left(\frac{1}{x}p + p\frac{1}{x}\right) \\ T_2 &= xp + px + \frac{2C}{A} \\ T_3 &= x^2 \end{aligned} \tag{7}$$

which are closed with respect to the generators of the algebra SU(1,1); i.e.,

$$\begin{aligned} [T_1, T_2] &= -4i\hbar T_1 \\ [T_2, T_3] &= -4i\hbar T_3 \\ [T_3, T_1] &= 2i\hbar T_2. \end{aligned} \tag{8}$$

We note that the above Lie algebra $\{T_1, T_2, T_3\}$ is identical to the oscillator algebra.

Now, we look for the invariant in the form

$$I(t) = \mu_1(t)T_1 + \mu_2(t)T_2 + \mu_3(t)T_3. \tag{9}$$

Using $\frac{\partial I}{\partial t} = \frac{i}{\hbar}\left[I, \widetilde{H}\right]$ and the fact that $\frac{E(t)}{A(t)}$ and $\frac{C(t)}{A(t)}$ are constant, we may obtain the μ_r coefficients of Equation (9) as

$$\begin{aligned} \dot{\mu}_1 &= 4(\mu_2 D - \mu_1 B) \\ \dot{\mu}_2 &= 2(\mu_3 D - \mu_1 A) \\ \dot{\mu}_3 &= 4(\mu_3 B - \mu_2 A) \end{aligned} \tag{10}$$

which can be simplified by setting $\mu_1 = \rho^2$ where ρ is the solution of the auxiliary equation

$$\ddot{\rho}(t) - \frac{\dot{A}}{A}\dot{\rho}(t) + 2\left(2AD + \frac{\dot{A}B}{A} - 2B^2 - \dot{B}\right)\rho(t) = 4EA\frac{1}{\rho^3(t)} \tag{11}$$

and

$$\begin{aligned} \mu_2 &= \frac{1}{2A}(2B\rho^2(t) - \dot{\rho}(t)\rho(t)), \\ \mu_3 &= \frac{1}{4A^2}(2B\rho(t) - \dot{\rho}(t))^2 + \frac{E}{A}\frac{1}{\rho^2(t)} \end{aligned} \tag{12}$$

Thus, the invariant can be written in the form

$$I(t) = \rho^2(t)\left(p^2 + \frac{E}{A}\frac{1}{x^2} + \frac{C}{A}\left(\frac{1}{x}p + p\frac{1}{x}\right)\right) + \left(\frac{1}{4A^2}\left(2B\rho(t) - \dot{\rho}(t)\right)^2 \right.$$
$$\left. + \frac{E}{A}\frac{1}{\rho^2(t)}\right)x^2 + \frac{1}{2A}\left(2B\rho^2(t) - \dot{\rho}(t)\rho(t)\right)\left(xp + px + \frac{2C}{A}\right) \quad (13)$$

Equation (13) agrees with h the invariant operator obtained in Ref. Choi (2003), if we set $\frac{C}{A}$ constant in Equation (22) of Ref. Choi (2003). Recall that $\frac{C}{A}$ is constant according to Equations (7)–(9), (11) and (13) of Ref. Choi (2003) even though it is not explicitly stated in Ref. Choi (2003).

According to the Lewis-Riesenfeld theory (Lewis and Riesenfeld, 1969), given a physical system that contains an invariant operator $I(t)$, the following results can be obtained:

(a) its eigenvalues λ_n are time-independent,

$$I\phi_n(x,t) = \lambda_n \phi_n(x,t), \quad (14)$$

(b) its eigenfunctions $\phi_n(x,t)$ depend on time.
When $\phi_n(x,t)$ are multiplied by suitable phases such as $\exp[i\alpha_n(t)]$, with $\alpha_n(t)$ verifying

$$\hbar\dot{\alpha}_n(t) = \left\langle \phi_n | i\hbar\frac{\partial}{\partial t} - \widetilde{H}(t) | \phi_n \right\rangle, \quad (15)$$

then, the wavefunctions $\varphi_n(x,t) = \exp[i\alpha_n(t)]\,\phi_n(x,t)$ evolve according to the time-dependent Shrodinger equation. The general solution $\varphi(x,t)$ can then be written as

$$\varphi(x,t) = \sum_n C_n e^{i\alpha_n(t)}\phi_n(x,t) \quad (16)$$

where C_n are arbitrary constant coefficients fixed by the initial conditions of the physical system.

Let us consider the unitary transformation

$$\Phi_n(x) = S(t)\phi_n(x,t) \quad (17)$$

where the time-dependent unitary transformation

$$S(t) = \exp\left(\frac{i\ln\rho(t)}{2\hbar}(xp + px)\right) \times \exp\left(\frac{i\mu_2}{2\hbar\mu_1}x^2\right) \quad (18)$$

changes x as

$$x \to SxS^{-1} = \rho x \quad (19)$$

and p as

$$p \to SpS^{-1} = \frac{p}{\rho} - \frac{\mu_2}{\mu_1}\rho x \tag{20}$$

and which when it acts on awave function in the x representation gives

$$S^{-1}(t)\phi_n(x,t) = \exp\left(-\frac{\ln\rho(t)}{2}\right) \times \exp\left(-\frac{i\mu_2}{2\hbar\mu_1}x^2\right)\phi_n\left(\frac{1}{\rho}x, t\right). \tag{21}$$

Under this unitary transformation the eigenvalue Equation (14) is mapped into ordinary one dimensional time-independent Schrödinger equation similar to the radial equation of a two-dimensional harmonic oscillator in presence of the Aharanov–Bohm effect (Aharonov and Bohm, 1959; Haggen, 1990)

$$\left(\frac{\partial^2}{\partial x^2} + a\frac{1}{x}\frac{\partial}{\partial x} - cx^2 - d\frac{1}{x^2} + \frac{\lambda_n}{\hbar^2}\right)\Phi_n(x) = 0 \tag{22}$$

where $a = \frac{2iC}{\hbar A}$, $c = \frac{E}{\hbar^2 A}$ and $d = \frac{E+i\hbar C}{\hbar^2 A}$.

If one define the new variable

$$y = x^2 \tag{23}$$

and express Φ_n as

$$\Phi_n(y) = y^k e^{qy}\chi_n(y) \tag{24}$$

where k and q are given by

$$k = \frac{1}{4} - \frac{iC}{2\hbar A} + \frac{1}{2}\sqrt{\frac{1}{4} + \frac{E}{\hbar^2 A} - \frac{C^2}{\hbar^2 A^2}}, \tag{25}$$

$$q = -\frac{1}{2\hbar}\sqrt{\frac{E}{A}} \tag{26}$$

then, the above eigenvalue Equation (22) takes the form of associated Laguerre polynomial equation

$$\xi\frac{\partial^2\chi_n(\xi)}{\partial\xi^2} + (1+m-\xi)\frac{\partial\chi_n(\xi)}{\partial\xi} + \frac{1}{2}\left(\frac{\lambda_n}{2\hbar}\sqrt{\frac{A}{E}} - m - 1\right)\chi_n(\xi) = 0 \tag{27}$$

where

$$\xi = \frac{1}{\hbar}\sqrt{\frac{E}{A}}y, \tag{28}$$

and

$$m = \sqrt{\frac{1}{4} + \frac{E}{\hbar^2 A} - \frac{C^2}{\hbar^2 A^2}} \tag{29}$$

We can express $\chi_n(\xi)$ in terms of Laguerre polynomial as

$$\chi_n(\xi) = L_n^m(\xi) \tag{30}$$

where

$$n = \frac{1}{2}\left(\frac{\lambda_n}{2\hbar}\sqrt{\frac{A}{E}} - m - 1\right) \tag{31}$$

and consequently the constant eigenvalue λ_n is exactly given by

$$\lambda_n = 2\hbar\sqrt{\frac{E}{A}}\,(2n + m + 1)\,. \tag{32}$$

Once again, these eigenvalues λ_n are similar to those found in Ref. Choi (2003) if one set $\frac{C}{A}$ constant in Equation (40) of Ref. Choi (2003). The normalized eigenfunctin Φ_n can be written as

$$\Phi_n(x) = \left[\frac{2\Gamma(n+1)}{\Gamma(n+m+1)}\left(\frac{1}{\hbar}\sqrt{\frac{E}{A}}\right)^{m+1}\right]^{\frac{1}{2}} \times x^{m+\frac{1}{2}-\frac{iC}{\hbar A}}$$
$$\times \exp\left(-\frac{1}{2\hbar}\sqrt{\frac{E}{A}}\right) x^2 \times L_n^m\left(\frac{1}{\hbar}\sqrt{\frac{E}{A}}x^2\right). \tag{33}$$

The complete normalized eigenfunction $I(t)$ is thus given by

$$\begin{aligned}\phi_n(x,t) &= S^{-1}\Phi_n(x)\\ &= \left[\frac{2\Gamma(n+1)}{\Gamma(n+m+1)}\left(\frac{1}{\hbar\rho^2}\sqrt{\frac{E}{A}}\right)^{m+1}\right]^{\frac{1}{2}}\left(\frac{1}{\rho}\right)^{-\frac{iC}{\hbar A}} \times x^{m+\frac{1}{2}-\frac{iC}{\hbar A}}\\ &\quad \times \exp\left\{-\frac{1}{4}\left[\frac{i}{\hbar A}\left(2B - \frac{\dot{\rho}}{\rho}\right) + \frac{2}{\hbar\rho^2}\sqrt{\frac{E}{A}}\right]x^2\right\} L_n^m\left(\frac{1}{\hbar\rho^2}\sqrt{\frac{E}{A}}x^2\right)\end{aligned} \tag{34}$$

It should be pointed out that the eigenfunction [Equation (34)] of the invariant operator seems to be similar to the Choi's eigenstate [Equation (41) of ref. Choi (2003)]. However they are not similar because of the different definition of the constant m . In the present paper m is defined by Equation (29) which is different from the m defined in Equation (36) of Ref. Choi (2003).

There remains the problem of finding the phases $\alpha_n(t)$ which satisfy the Equation (15). By inserting Equation (34) into Equation(15), we can find

$$\alpha_n(t) = -2\,(2n + m + 1)\int_0^t \frac{\sqrt{E(t')A(t')}}{\rho^2(t')}dt' - \frac{C}{\hbar A}\int_0^t \frac{\dot{\rho}(t')}{\rho(t')}dt'. \tag{35}$$

Therefore the exact solution of the original Schrödinger Equation (1) associated to the Hamiltonian $H(x, p, t)$ (2), can now be found by combing the above results. We finally obtain

$$\psi_n(x,t) = U(t)\varphi_n(x,t) = \left[\frac{2\Gamma(n+1)}{\Gamma(n+m+1)}\left(\frac{1}{\hbar\rho^2}\sqrt{\frac{E}{A}}\right)^{m+1}\right]^{\frac{1}{2}}$$

$$\times\left(\frac{1}{\rho(0)}\right)^{-\frac{iC}{\hbar A}} x^{m+\frac{1}{2}-\frac{iC}{\hbar A}} \exp\left\{-\frac{1}{4}\left[\frac{i}{\hbar A}\left(2B-\frac{\dot{\rho}}{\rho}\right)+\frac{2}{\hbar\rho^2}\sqrt{\frac{E}{A}}\right]x^2\right\}$$

$$\times L_n^m\left(\frac{1}{\hbar\rho^2}\sqrt{\frac{E}{A}}x^2\right)\times\exp\left\{-2i\,(2n+m+1)\int_0^t\frac{\sqrt{E(t')A(t')}}{\rho^2(t')}dt'\right.$$

$$\left.+\frac{i}{\hbar}\int_0^t\frac{2CB}{A}dt'\right\}. \tag{36}$$

where the global phase is given by

$$\theta_n(t) = -2\,(2n+m+1)\int_0^t\frac{\sqrt{E(t')A(t')}}{\rho^2(t')}dt' + \frac{i}{\hbar}\int_0^t\frac{2CB}{A}dt' \tag{37}$$

Note that the wave function found here (Equation (36)) and the global phase (Equation (37)) are different from those derived in Ref. Choi (2003) even if $\frac{C}{A}$ is set constant in Equation (46) of Ref. Choi (2003).

The expressions (36) for $\psi_n(x,t)$ recovers ones for the particular cases $C = 0$, considered previously (Pedrosa *et al.*, 1997; Trifonov, 1999). While for this particular case ($C = 0$) the Choi wave function failed to recover the wave function of (Pedrosa *et al.*, 1997; Trifonov, 1999) (there is an extra factor $\frac{1}{2}$ in the phase); to check this, one has to substitute $A \to \frac{A}{2}$ and $E \to \frac{E}{2}$ in the phase (Equation (45)) of Ref. Choi (2003).

For $B = C = \dot{A} = \dot{E} = 0$ the wave function $\psi_n(x,t)$ Equation (36) coincide with those found in (Dodonov *et al.*, 1998).

In summary, we give an alternative and correct approach to find the wave function of a system described by the Hamiltonian (Equation (2)). This simple method is based on dynamical invariant and SU(1,1) algebra. We believe that the present method gives the correct result compared to a recently published one.

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